A UNIFIED GENERALIZATION OF SOME QUADRATURE RULES AND ERROR BOUNDS

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Abstract. By introducing a parameter, we give a unified generalization of some quadrature rules, which not only unify the recent results about error bounds for generalized mid-point, trapezoid and Simpson's rules, but also give some new error bounds for other quadrature rules as special cases. Especially, two sharp error inequalities are derived when n is an odd and an even integer, respectively.

1. Introduction

Error analysis for known and new quadrature rules has been extensively studied in recent years. The approach from an inequalities point of view to estimate the error terms has been used in these studies (see [1]-[22] and the references therein).

In [22], by appropriately choosing the Peano kernel

$$T_n(x)=\begin{cases}\dfrac{(x-a)^{n-1}}{n!}\left[x+\dfrac{(n-2)a-nb}{2}\right], & x\in\left[a,\dfrac{a+b}{2}\right],\\ \dfrac{(x-b)^{n-1}}{n!}\left[x+\dfrac{(n-2)b-na}{2}\right], & x\in\left(\dfrac{a+b}{2},b\right],\end{cases}\tag{1}$$

error inequalities for a generalized trapezoid rule were given as follows.

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a function such that $f^{(n-1)}$, $n>1$, is absolutely continuous.*

If there exist real numbers γ_n,Γ_n such that $\gamma_n\le f^{(n)}(x)\le\Gamma_n$, $x\in[a,b]$, then

$$\begin{aligned}&\left|\int_a^b f(x)dx-\frac{f(a)+f(b)}{2}(b-a)+\sum_{i=1}^{\lfloor\frac{n-1}{2}\rfloor}\frac{2i(b-a)^{2i+1}}{2^{2i}(2i+1)!}f^{(2i)}\left(\frac{a+b}{2}\right)\right|\\&\le\frac{\Gamma_n-\gamma_n}{(n+1)!}\frac{n}{2^{n+1}}(b-a)^{n+1},\quad \text{if } n \text{ is odd}\end{aligned}\tag{2}$$

and

$$\begin{aligned}&\left|\int_a^b f(x)dx-\frac{f(a)+f(b)}{2}(b-a)+\sum_{i=1}^{\lfloor\frac{n-1}{2}\rfloor}\frac{2i(b-a)^{2i+1}}{2^{2i}(2i+1)!}f^{(2i)}\left(\frac{a+b}{2}\right)\right|\\&\le\frac{(b-a)^{n+1}n}{(n+1)!\,2^n}\|f^{(n)}\|_\infty,\quad \text{if } n \text{ is even}\end{aligned}\tag{3}$$

where $\lfloor\frac{n-1}{2}\rfloor$ denotes the integer part of $\frac{n-1}{2}$.

If there exists a real number γ_n such that $\gamma_n\le f^{(n)}(x)$, $x\in[a,b]$, then

$$\begin{aligned}&\left|\int_a^b f(x)dx-\frac{f(a)+f(b)}{2}(b-a)+\sum_{i=1}^{\lfloor\frac{n-1}{2}\rfloor}\frac{2i(b-a)^{2i+1}}{2^{2i}(2i+1)!}f^{(2i)}\left(\frac{a+b}{2}\right)\right|\\&\le\left[\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}-\gamma_n\right]\frac{n-1}{n!\,2^n}(b-a)^{n+1},\quad \text{if } n \text{ is odd.}\end{aligned}\tag{4}$$

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If there exists a real number Γ_n such that $f^{(n)}(x) \leq \Gamma_n$, $x \in [a,b]$, then

$$\begin{aligned}&\left|\int_a^b f(x)dx - \frac{f(a)+f(b)}{2}(b-a) + \sum_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} \frac{2i(b-a)^{2i+1}}{2^{2i}(2i+1)!} f^{(2i)}\left(\frac{a+b}{2}\right)\right| \\ &\leq \left[\Gamma_n - \frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}\right] \frac{n-1}{n!\,2^n}(b-a)^{n+1}, \quad \text{if } n \text{ is odd.} \end{aligned} \tag{5}$$

In [14], by choosing

$$M_n(x) = \begin{cases} \dfrac{(x-a)^n}{n!}, & x \in \left[a, \dfrac{a+b}{2}\right], \\ \dfrac{(x-b)^n}{n!}, & x \in \left(\dfrac{a+b}{2}, b\right], \end{cases} \tag{6}$$

Liu provided the following sharp perturbed midpoint inequalities by using the variant of the Grüss inequality.

Theorem 2. *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n)}$ is integrable with $\gamma_n \leq f^{(n)}(x) \leq \Gamma_n$ for all $x \in [a,b]$, where $\gamma_n, \Gamma_n \in R$ are constants. Then*

$$\begin{aligned}&\left|\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)(b-a) - \sum_{k=1}^{n-1} \frac{[1+(-1)^k](b-a)^{k+1}}{2^{k+1}(k+1)!} f^{(k)}\left(\frac{a+b}{2}\right)\right. \\ &\left. - \frac{[1+(-1)^n](b-a)^{n+1}}{2^{n+1}(n+1)!} \frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}\right| \\ &\leq \frac{\Gamma_n - \gamma_n}{(n+1)!}\left[\frac{1-(-1)^n}{2} + \frac{[1+(-1)^n]n}{(n+1)\sqrt[n]{n+1}}\right] \frac{1}{2^{n+1}}(b-a)^{n+1}. \end{aligned} \tag{7}$$

In [13], by choosing the kernel

$$S_n(x) = \begin{cases} \dfrac{(x-a)^{n-1}}{n!}\left[x-a-\dfrac{n(b-a)}{6}\right], & x \in \left[a, \dfrac{a+b}{2}\right], \\ \dfrac{(x-b)^{n-1}}{n!}\left[x-b+\dfrac{n(b-a)}{6}\right], & x \in \left(\dfrac{a+b}{2}, b\right], \end{cases} \tag{8}$$

an inequality of Simpson type for an n-times continuously differentiable mapping was given as follows.

Theorem 3. *Let $f : [a,b] \to \mathbb{R}$ be an n-times continuously differentiable mapping, $n \geq 1$ and such that $\|f^{(n)}\|_\infty := \sup_{x\in(a,b)} |f^{(n)}(x)| < \infty$. Then*

$$\begin{aligned}&\left|\int_a^b f(x)dx - \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] + \sum_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} \frac{(i-1)(b-a)^{2i+1}}{3(2i+1)!2^{2i-1}} f^{(2i)}\left(\frac{a+b}{2}\right)\right| \\ &\leq \frac{(b-a)^{n+1}}{(n+1)!}\|f^{(n)}\|_\infty \times \begin{cases} \dfrac{4n^n}{6^{n+1}} - \dfrac{n-2}{3\cdot 2^n}, & \text{if } n < 3, \\ \dfrac{n-2}{3\cdot 2^n}, & \text{if } n \geq 3. \end{cases} \end{aligned} \tag{9}$$

In [17], using the well-known pre-Grüss inequality, Pečarić and Varošanec obtained different error bounds for inequality (9). Liu [15] generalized inequality (9) and also provided an improvement of [17]. More recently, Shi and Liu [19] further derived some sharp Simpson type inequalities.

The purpose of this paper is to give a unified generalization of some quadrature rules, which not only unify the above results about error bounds for generalized mid-point, trapezoid and Simpson's rules, but also give some new error bounds for other quadrature rules as special cases. Especially, we will derive two sharp error inequalities when n is an odd and an even integer, respectively.

2. Preliminaries

In this section we present some lemmas and notations needed in the proof of our main results.

Lemma 1. *Let $f : [a, b] \to \mathbb{R}$ be a function such that $f^{(n-1)}$, $n \geq 1$, is absolutely continuous. Then*

$$\begin{aligned}\int_a^b f(x)dx =&(b-a)\left[(1-\theta)f\left(\frac{a+b}{2}\right)+\theta\frac{f(a)+f(b)}{2}\right]\\&+\sum_{i=1}^{\lfloor\frac{n-1}{2}\rfloor}\frac{[1-\theta(2i+1)](b-a)^{2i+1}}{(2i+1)!2^{2i}}f^{(2i)}\left(\frac{a+b}{2}\right)+R(f),\end{aligned}\tag{10}$$

for all $\theta \in [0, 1]$, where

$$R(f) = (-1)^n \int_a^b G_n(x)f^{(n)}(x)dx,$$

and

$$G_n(x) = \begin{cases} \dfrac{(x-a)^{n-1}}{n!}\left[x-a-\dfrac{\theta n(b-a)}{2}\right], & x \in \left[a, \dfrac{a+b}{2}\right], \\ \dfrac{(x-b)^{n-1}}{n!}\left[x-b+\dfrac{\theta n(b-a)}{2}\right], & x \in \left(\dfrac{a+b}{2}, b\right]. \end{cases}\tag{11}$$

Proof. We briefly sketch the proof. We introduce the notations

$$P_n(x) = \frac{(x-a)^{n-1}}{n!}\left[x-a-\frac{\theta n(b-a)}{2}\right],$$

$$Q_n(x) = \frac{(x-b)^{n-1}}{n!}\left[x-b+\frac{\theta n(b-a)}{2}\right],$$

then one can see that P_n and Q_n form Appell sequences of polynomials, that is

$$P_n'(x) = P_{n-1}(x), \quad Q_n'(x) = Q_{n-1}(x), \quad P_0(x) = Q_0(x) = 1.$$

Thus one can use integration by parts to prove that (10) holds. □

Lemma 2. *The Peano kernels $G_n(t)$, satisfies*

$$\int_a^b G_n(x)dx = \begin{cases} 0, & n \text{ odd}, \\ \dfrac{(b-a)^{n+1}}{n!\,2^n}\left(\dfrac{1}{n+1}-\theta\right), & n \text{ even}, \end{cases}\tag{12}$$

$$\int_a^b |G_n(x)|dx = \begin{cases} \dfrac{(b-a)^{n+1}}{n!\,2^n}\left(\theta-\dfrac{1}{n+1}\right), & \text{if } \theta n \geq 1, \\ \dfrac{(b-a)^{n+1}}{n(n+1)!\,2^n}\left[2(\theta n)^{n+1}-\theta n(n+1)+n\right], & \text{if } 1 \leq \theta n < 1, \end{cases}\tag{13}$$

$$\max_{x\in[a,b]}|G_n(x)| = \begin{cases} \dfrac{(\theta n-1)(b-a)^n}{n!\,2^n}, & \text{if } \theta n > \theta+1, \\ \dfrac{\theta^n(n-1)^{n-1}(b-a)^n}{n!\,2^n}, & \text{if } 1 \leq \theta n \leq \theta+1 \text{ and } n > 1, \\ \max\{1-\theta n, \theta^n(n-1)^{n-1}\}\dfrac{(b-a)^n}{n!\,2^n}, & \text{if } \theta n < 1 \text{ and } n > 1, \\ \max\{1-\theta, \theta\}\dfrac{b-a}{2}, & \text{if } n = 1, \end{cases}\tag{14}$$

$$\int_a^b G_n^2(x)dx = \frac{[\theta^2n^2(2n+1)-\theta(4n^2-1)+(2n-1)](b-a)^{2n+1}}{(2n+1)(2n-1)(n!)^2 2^{2n}},\tag{15}$$

and

$$
\begin{aligned}
&\max_{x\in[a,b]}\left|G_{2m}(x)-\frac{1}{b-a}\int_a^b G_{2m}(x)dx\right| \\
&=\begin{cases}
\max\left\{\theta-\dfrac{1}{2m+1},\left[\theta(2m-1)-\dfrac{2m}{2m+1}\right]\right\}\dfrac{(b-a)^{2m}}{(2m)!\,2^{2m}}, & \text{if } \theta(2m-1)\ge 1,\\
\max\left\{\left|\theta-\dfrac{1}{2m+1}\right|,\left|\theta(2m-1)-\dfrac{2m}{2m+1}\right|,\right. & \\
\qquad\left.\left|\theta-\dfrac{1}{2m+1}-\theta^{2m}(2m-1)^{2m-1}\right|\right\}\dfrac{(b-a)^{2m}}{(2m)!\,2^{2m}}, & \text{if } \theta(2m-1)<1.
\end{cases}
\end{aligned}
\tag{16}
$$

Proof. A simple calculation gives

$$\int_a^b G_n(x)dx=\frac{(b-a)^{n+1}}{n!\,2^{n+1}}\left(\frac{1}{n+1}-\theta\right)[1-(-1)^{n+1}],$$

from which we see that (12) holds.

We have

$$\int_a^b|G_n(x)|dx=\int_a^{\frac{a+b}{2}}|P_n(x)|dx+\int_{\frac{a+b}{2}}^b|Q_n(x)|dx=2\int_a^{\frac{a+b}{2}}|P_n(x)|dx=\frac{(b-a)^{n+1}}{n!\,2^n}\int_0^1|t^{n-1}(t-\theta n)|dt,$$

by substitution $x=a+\frac{b-a}{2}t$. If $\theta n\ge 1$, one has

$$\int_a^b|G_n(x)|dx=\frac{(b-a)^{n+1}}{n!\,2^n}\left(\theta n\int_0^1 t^{n-1}dt-\int_0^1 t^n dt\right)=\frac{(b-a)^{n+1}}{n!\,2^n}\left(\theta-\frac{1}{n+1}\right).$$

If $0\le\theta n<1$, one gets

$$
\begin{aligned}
\int_a^b|G_n(x)|dx=&\frac{(b-a)^{n+1}}{n!\,2^n}\left[\int_0^{\theta n}t^{n-1}(\theta n-t)dt+\int_{\theta n}^1 t^{n-1}(t-\theta n)dt\right]\\
=&\frac{(b-a)^{n+1}}{n(n+1)!\,2^n}\left[2(\theta n)^{n+1}-\theta n(n+1)+n\right].
\end{aligned}
$$

By combining the above two cases, (13) is established.

We have

$$\max_{x\in[a,b]}|G_n(x)|=\max\left\{\max_{x\in\left[a,\frac{a+b}{2}\right]}|P_n(x)|,\max_{x\in\left[\frac{a+b}{2},b\right]}|Q_n(x)|\right\}=\max_{x\in\left[a,\frac{a+b}{2}\right]}|P_n(x)|.$$

When $n=1$,

$$P_1(x)=x-a-\frac{\theta(b-a)}{2},\quad x\in\left[a,\frac{a+b}{2}\right],$$

then

$$\max_{x\in[a,b]}|G_1(x)|=\max\{1-\theta,\theta\}\frac{b-a}{2}.$$

When $n>1$, since $P_n'(x)=0$ gives $x=a$ or $x=a+\frac{\theta(n-1)(b-a)}{2}$, we divide the proof of (14) into three steps according to the different intervals of θn.

Case $\theta n>\theta+1$: We have

$$a<\frac{a+b}{2}<a+\frac{\theta(n-1)(b-a)}{2}<a+\frac{\theta n(b-a)}{2}.$$

So, we can get

$$P_n(x)<0,\quad P_n(x)\text{ is decreasing, for } x\in\left[a,\frac{a+b}{2}\right].$$

Therefore,

$$\max_{x\in[a,b]}|G_n(x)|=-\min_{x\in\left[a,\frac{a+b}{2}\right]}P_n(x)=-P_n\left(\frac{a+b}{2}\right)=\frac{(\theta n-1)(b-a)^n}{n!\,2^n}.$$

Case $1 \le \theta n \le \theta + 1$: We have

$$a < a + \frac{\theta(n-1)(b-a)}{2} \le \frac{a+b}{2} \le a + \frac{\theta n(b-a)}{2}.$$

So, we can get

$$P_n(x) < 0, \quad \text{for } x \in \left[a, \frac{a+b}{2}\right].$$

Therefore,

$$\max_{x\in[a,b]} |G_n(x)| = -\min_{x\in\left[a,\frac{a+b}{2}\right]} P_n(x) = -P_n\left(a + \frac{\theta(n-1)(b-a)}{2}\right) = \frac{\theta^n (n-1)^{n-1}(b-a)^n}{n!\,2^n}.$$

Case $\theta n < 1$: We have

$$a < a + \frac{\theta(n-1)(b-a)}{2} < a + \frac{\theta n(b-a)}{2} < \frac{a+b}{2}.$$

So, we can get

$$P_n(x) < 0, \quad \text{for } x \in \left[a, a + \frac{\theta n(b-a)}{2}\right); \quad P_n(x) \ge 0, \quad \text{for } x \in \left[a + \frac{\theta n(b-a)}{2}, \frac{a+b}{2}\right].$$

Therefore,

$$\max_{x\in[a,b]} |G_n(x)| = \max\left\{\left|P_n\left(\frac{a+b}{2}\right)\right|, \left|P_n\left(a + \frac{\theta(n-1)(b-a)}{2}\right)\right|\right\} = \max\{1-\theta n, \theta^n(n-1)^{n-1}\}\frac{(b-a)^n}{n!\,2^n}.$$

By combining the above three cases, (14) is established.

(15) can be obtained by a direct calculation.

From (12), we have

$$\begin{aligned}
&\max_{x\in[a,b]} \left|G_{2m}(x) - \frac{1}{b-a}\int_a^b G_{2m}(x)dx\right| \\
&= \max_{x\in[a,b]} \left|G_{2m}(x) - \frac{(b-a)^{2m}}{(2m)!\,2^{2m}}\left(\frac{1}{2m+1} - \theta\right)\right| \\
&= \max_{x\in\left[a,\frac{a+b}{2}\right]} \left|\frac{(x-a)^{2m-1}}{(2m)!}\left[x - a - \frac{2\theta m(b-a)}{2}\right] - \frac{(b-a)^{2m}}{(2m)!\,2^{2m}}\left(\frac{1}{2m+1} - \theta\right)\right| := \max_{x\in\left[a,\frac{a+b}{2}\right]} |F_n(x)|
\end{aligned}$$

and

$$F_n(a) = \frac{(b-a)^{2m}}{(2m)!\,2^{2m}}\left(\theta - \frac{1}{2m+1}\right).$$

We divide the proof of (16) into two steps according to the different intervals of $\theta(2m-1)$.

Case $\theta(2m-1) \ge 1$: We have

$$F_n(a) > 0, \quad a + \frac{\theta(2m-1)(b-a)}{2} \ge \frac{a+b}{2}.$$

Thus, we get

$$\max_{x\in\left[a,\frac{a+b}{2}\right]} |F_n(x)| = \max\left\{F_n(a), \left|F_n\left(\frac{a+b}{2}\right)\right|\right\} = \max\left\{\theta - \frac{1}{2m+1}, \left[\theta(2m-1) - \frac{2m}{2m+1}\right]\right\}\frac{(b-a)^{2m}}{(2m)!\,2^{2m}}.$$

Case $\theta(2m-1) < 1$: We have

$$a < a + \frac{\theta(2m-1)(b-a)}{2} < \frac{a+b}{2}.$$

Thus, we obtain

$$\begin{aligned}
&\max_{x\in\left[a,\frac{a+b}{2}\right]} |F_n(x)| = \max\left\{|F_n(a)|, \left|F_n\left(\frac{a+b}{2}\right)\right|, \left|F_n\left(a + \frac{\theta(2m-1)(b-a)}{2}\right)\right|\right\} \\
&= \max\left\{\left|\theta - \frac{1}{2m+1}\right|, \left|\theta(2m-1) - \frac{2m}{2m+1}\right|, \left|\theta - \frac{1}{2m+1} - \theta^{2m}(2m-1)^{2m-1}\right|\right\}\frac{(b-a)^{2m}}{(2m)!\,2^{2m}}.
\end{aligned}$$

By combining the above two cases, (16) is established. $\square$

Before we end this section, we introduce the notations

$$I = \int_a^b f(x)dx,$$

$$F_n = (b-a)\left[(1-\theta)f\left(\frac{a+b}{2}\right) + \theta\frac{f(a)+f(b)}{2}\right] + \sum_{i=2}^{\lfloor\frac{n-1}{2}\rfloor} \frac{[1-\theta(2i+1)](b-a)^{2i+1}}{(2i+1)!2^{2i}} f^{(2i)}\left(\frac{a+b}{2}\right).$$

3. Main results

We first establish three error inequalities for $f^{(n)} \in L^1[a,b]$, $L^2[a,b]$ and $L^\infty[a,b]$, respectively.

Theorem 4. *Let $f : [a,b] \to \mathbb{R}$ be a function such that $f^{(n-1)}$, $n > 1$, is absolutely continuous on $[a,b]$. If $f^{(n)} \in L^1[a,b]$, then we have*

$$|I - F_n| \le \frac{(b-a)^n}{n!\,2^n}\|f^{(n)}\|_1 \times \begin{cases} \theta n - 1, & \text{if } \theta n > \theta + 1, \\ \theta^n (n-1)^{n-1}, & \text{if } 1 < \theta n \le \theta + 1, \\ \max\{1-\theta n, \theta^n (n-1)^{n-1}\}, & \text{if } \theta n \le 1, \end{cases} \tag{17}$$

for all $\theta \in [0,1]$, where $\|f^{(n)}\|_1 := \int_a^b |f^{(n)}(x)|dx$ is the usual Lebesgue norm on $L^1[a,b]$.

Proof. By using the identity (10), we have

$$|I - F_n| = \left|\int_a^b G_n(x) f^{(n)}(x)dx\right| \le \max_{x\in[a,b]} |G_n(x)| \int_a^b |f^{(n)}(x)|dx. \tag{18}$$

Consequently, inequality (17) follows from (18) and (14). □

Remark 1. If we take $\theta = 1$ in (17), we get the trapezoid type inequality

$$\left|\int_a^b f(x)dx - \frac{f(a)+f(b)}{2}(b-a) + \sum_{i=1}^{\lfloor\frac{n-1}{2}\rfloor} \frac{2i(b-a)^{2i+1}}{2^{2i}(2i+1)!} f^{(2i)}\left(\frac{a+b}{2}\right)\right|$$
$$\le \frac{(b-a)^n}{n!\,2^n}\|f^{(n)}\|_1 \times \begin{cases} n-1, & \text{if } n > 1, \\ 1, & \text{if } n = 1. \end{cases}$$

If we take $\theta = \frac{1}{3}$ in (17), we recapture the Simpson type inequality

$$\left|\int_a^b f(x)dx - \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] + \sum_{i=1}^{\lfloor\frac{n-1}{2}\rfloor} \frac{(i-1)(b-a)^{2i+1}}{3(2i+1)!2^{2i-1}} f^{(2i)}\left(\frac{a+b}{2}\right)\right|$$
$$\le \|f^{(n)}\|_1 \times \begin{cases} \dfrac{(n-3)(b-a)^n}{3(n!)2^n}, & \text{if } n \ge 4, \\ \dfrac{(b-a)^3}{324}, & \text{if } n = 3, \\ \dfrac{(b-a)^2}{24}, & \text{if } n = 2, \\ \dfrac{b-a}{3}, & \text{if } n = 1, \end{cases}$$

which has been appeared in [15, Theorem 4].

If we take $\theta = 0$ in (17), we recapture the midpoint type inequality

$$\left|\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)(b-a) - \sum_{i=1}^{\lfloor\frac{n-1}{2}\rfloor} \frac{(b-a)^{2i+1}}{(2i+1)!2^{2i}} f^{(2i)}\left(\frac{a+b}{2}\right)\right| \le \frac{(b-a)^n}{n!2^n}\|f^{(n)}\|_1,$$

which has been appeared in [3, Corollary 4.15].

If we take $\theta = \frac{1}{2}$ in (17), we get the averaged midpoint-trapezoid type inequality

$$\left| \int_a^b f(x)dx - \frac{b-a}{4}\left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b)\right] + \sum_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} \frac{(2i-1)(b-a)^{2i+1}}{(2i+1)!2^{2i+1}} f^{(2i)}\left(\frac{a+b}{2}\right)\right|$$
$$\leq \|f^{(n)}\|_1 \times \begin{cases} \dfrac{(n-2)(b-a)^n}{n!2^{n+1}}, & \text{if } n \geq 3, \\ \dfrac{(b-a)^2}{32}, & \text{if } n = 2, \\ \dfrac{b-a}{4}, & \text{if } n = 1. \end{cases}$$

Theorem 5. *Let $f : [a, b] \to \mathbb{R}$ be a function such that $f^{(n-1)}$, $n > 1$, is absolutely continuous on $[a, b]$. If $f^{(n)} \in L^2[a, b]$, then we have*

$$|I - F_n| \leq \frac{(b-a)^{n+\frac{1}{2}}}{n!\, 2^n} \|f^{(n)}\|_2 \sqrt{\frac{\theta^2 n^2(2n+1) - \theta(4n^2-1) + (2n-1)}{(2n+1)(2n-1)}}, \tag{19}$$

for all $\theta \in [0, 1]$, where $\|f^{(n)}\|_2 := \left(\int_a^b |f^{(n)}(x)|^2 dx\right)^{\frac{1}{2}}$ is the usual Lebesgue norm on $L^2[a, b]$.

Proof. By using the identity (10), we have

$$|I - F_n| = \left| \int_a^b G_n(x) f^{(n)}(x)dx \right| \leq \|f^{(n)}\|_2 \|G_n\|_2. \tag{20}$$

Consequently, inequality (19) follows from (20) and (15). □

Remark 2. If we take $\theta = 1$ in (19), we get the trapezoid type inequality

$$\left| \int_a^b f(x)dx - \frac{f(a) + f(b)}{2}(b-a) + \sum_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} \frac{2i(b-a)^{2i+1}}{2^{2i}(2i+1)!} f^{(2i)}\left(\frac{a+b}{2}\right)\right|$$
$$\leq \frac{(b-a)^{n+\frac{1}{2}}}{n!\, 2^n} \|f^{(n)}\|_2 \sqrt{\frac{n(2n^2-3n+2)}{(2n+1)(2n-1)}}.$$

If we take $\theta = \frac{1}{3}$ in (19), we recapture the Simpson type inequality

$$\left| \int_a^b f(x)dx - \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] + \sum_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} \frac{(i-1)(b-a)^{2i+1}}{3(2i+1)!2^{2i-1}} f^{(2i)}\left(\frac{a+b}{2}\right)\right|$$
$$\leq \frac{(b-a)^{n+\frac{1}{2}}}{n!\, 2^n} \|f^{(n)}\|_2 \sqrt{\frac{2n^3 - 11n^2 + 18n - 6}{9(2n+1)(2n-1)}},$$

which has been appeared in [15, Theorem 5].

If we take $\theta = 0$ in (19), we recapture the midpoint type inequality

$$\left| \int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)(b-a) - \sum_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} \frac{(b-a)^{2i+1}}{(2i+1)!2^{2i}} f^{(2i)}\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)^{n+\frac{1}{2}}}{n!\, 2^n} \|f^{(n)}\|_2 \frac{1}{\sqrt{2n+1}},$$

which has been appeared in [3, Corollary 4.15].

If we take $\theta = \frac{1}{2}$ in (19), we get the averaged midpoint-trapezoid type inequality

$$\left| \int_a^b f(x)dx - \frac{b-a}{4}\left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b)\right] + \sum_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} \frac{(2i-1)(b-a)^{2i+1}}{(2i+1)!2^{2i+1}} f^{(2i)}\left(\frac{a+b}{2}\right)\right|$$
$$\leq \frac{(b-a)^{n+\frac{1}{2}}}{n!\, 2^n} \|f^{(n)}\|_2 \sqrt{\frac{2n^3 - 7n^2 + 8n - 2}{4(2n+1)(2n-1)}}.$$

Theorem 6. *Let $f:[a,b]\to\mathbb{R}$ be an n-times continuously differentiable mapping, $n\geq 1$ and such that $\|f^{(n)}\|_\infty<\infty$. Then we have*

$$|I-F_n|\leq \frac{(b-a)^{n+1}}{(n+1)!2^n}\|f^{(n)}\|_\infty\times\begin{cases}\theta(n+1)-1, & \text{if } \theta n\geq 1,\\ 2\theta^{n+1}n^n-\theta(n+1)+1, & \text{if } 0\leq\theta n<1,\end{cases}\tag{21}$$

for all $\theta\in[0,1]$.

Proof. Using the identity (10), we get

$$|I-F_n|=\left|\int_a^b G_n(x)f^{(n)}(x)dx\right|\leq\|f^{(n)}\|_\infty\int_a^b|G_n(x)|dx.\tag{22}$$

Consequently, inequality (21) follows from (22) and (13). □

Remark 3. If we take $\theta=1$ and $\theta=\frac{1}{3}$ in (21), respectively, we recapture the trapezoid type inequality (3) and the Simpson type inequality (9), respectively. If we take $\theta=0$ in (21), we recapture the midpoint type inequality appeared in [3, Corollary 4.15].

Next, if $f^{(n)}$ is integrable and bounded, we prove some new error inequalities and perturbed error inequalities, respectively.

Theorem 7. *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n)}(n\geq 1)$ is integrable with $\gamma_n\leq f^{(n)}(x)\leq\Gamma_n$ for all $x\in[a,b]$, where $\gamma_n,\Gamma_n\in R$ are constants.*

(1) If n is an odd integer, we have

$$|I-F_n|\leq\frac{\Gamma_n-\gamma_n}{2}\frac{(b-a)^{n+1}}{(n+1)!2^n}\times\begin{cases}\theta(n+1)-1, & \text{if } \theta n\geq 1,\\ 2\theta^{n+1}n^n-\theta(n+1)+1, & \text{if } 0\leq\theta n<1,\end{cases}\tag{23}$$

$$\begin{aligned}|I-F_n|&\leq\left[\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}-\gamma_n\right]\frac{(b-a)^{n+1}}{n!2^n}\\&\quad\times\begin{cases}\theta n-1, & \text{if } \theta n>\theta+1,\\ \theta^n(n-1)^{n-1}, & \text{if } 1<\theta n\leq\theta+1,\\ \max\{1-\theta n,\theta^n(n-1)^{n-1}\}, & \text{if } \theta n\leq 1,\end{cases}\end{aligned}\tag{24}$$

and

$$\begin{aligned}|I-F_n|&\leq\left[\Gamma_n-\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}\right]\frac{(b-a)^{n+1}}{n!2^n}\\&\quad\times\begin{cases}\theta n-1, & \text{if } \theta n>\theta+1,\\ \theta^n(n-1)^{n-1}, & \text{if } 1<\theta n\leq\theta+1,\\ \max\{1-\theta n,\theta^n(n-1)^{n-1}\}, & \text{if } \theta n\leq 1,\end{cases}\end{aligned}\tag{25}$$

for all $\theta\in[0,1]$.

(2) If n is an even integer ($n=2m$), we have

$$\begin{aligned}&\left|I-F_{2m}-\frac{(b-a)^{2m+1}}{(2m)!\,2^{2m}}\left(\frac{1}{2m+1}-\theta\right)\frac{f^{(2m-1)}(b)-f^{(2m-1)}(a)}{b-a}\right|\\&\leq\left[\frac{f^{(2m-1)}(b)-f^{(2m-1)}(a)}{b-a}-\gamma_{2m}\right]\frac{(b-a)^{2m+1}}{(2m)!\,2^{2m}}\\&\quad\times\begin{cases}\max\left\{\theta-\dfrac{1}{2m+1},\left[\theta(2m-1)-\dfrac{2m}{2m+1}\right]\right\}, & \text{if } \theta(2m-1)\geq 1,\\ \max\left\{\left|\theta-\dfrac{1}{2m+1}\right|,\left|\theta(2m-1)-\dfrac{2m}{2m+1}\right|,\right.\\ \qquad\left.\left|\theta-\dfrac{1}{2m+1}-\theta^{2m}(2m-1)^{2m-1}\right|\right\}, & \text{if } \theta(2m-1)<1,\end{cases}\end{aligned}\tag{26}$$

and

$$\left|I - F_{2m} - \frac{(b-a)^{2m+1}}{(2m)!\,2^{2m}}\left(\frac{1}{2m+1} - \theta\right)\frac{f^{(2m-1)}(b) - f^{(2m-1)}(a)}{b-a}\right|$$
$$\leq \left[\Gamma_{2m} - \frac{f^{(2m-1)}(b) - f^{(2m-1)}(a)}{b-a}\right]\frac{(b-a)^{2m+1}}{(2m)!\,2^{2m}}$$
$$\times \begin{cases} \max\left\{\theta - \frac{1}{2m+1}, \left[\theta(2m-1) - \frac{2m}{2m+1}\right]\right\}, & \text{if } \theta(2m-1) \geq 1, \\ \max\left\{\left|\theta - \frac{1}{2m+1}\right|, \left|\theta(2m-1) - \frac{2m}{2m+1}\right|, \right. & \\ \qquad \left.\left|\theta - \frac{1}{2m+1} - \theta^{2m}(2m-1)^{2m-1}\right|\right\}, & \text{if } \theta(2m-1) < 1, \end{cases} \tag{27}$$

for all $\theta \in [0,1]$.

Proof. (1) For n odd, by (12) and (10), we get

$$I - F_n = -\int_a^b G_n(x)[f^{(n)}(x) - C]dx, \tag{28}$$

where $C \in R$ is a constant.

If we choose $C = \frac{\gamma_n + \Gamma_n}{2}$, we have

$$|I - F_n| \leq \max_{x\in[a,b]}\left|f^{(n)}(x) - \frac{\gamma_n + \Gamma_n}{2}\right|\int_a^b |G_n(x)|dx = \frac{\Gamma_n - \gamma_n}{2}\int_a^b |G_n(x)|dx, \tag{29}$$

and hence inequality (23) follows from (29) and (13).

If we choose $C = \gamma_n$, we have

$$|I - F_n| \leq \max_{x\in[a,b]}|G_n(x)|\int_a^b |f^{(n)}(x) - \gamma_n|dx, \tag{30}$$

and hence inequality (24) follows from (30) and (14).

Similarly we can prove that inequality (25) holds.

(2) For n even, by (12) and (10), we can obtain

$$\left|I - F_{2m} - \frac{(b-a)^{2m+1}}{(2m)!\,2^{2m}}\left(\frac{1}{2m+1} - \theta\right)\frac{f^{(2m-1)}(b) - f^{(2m-1)}(a)}{b-a}\right|$$
$$= \left|\int_a^b \left[G_{2m}(x) - \frac{1}{b-a}\int_a^b G_{2m}(x)dx\right][f^{(2m)}(x) - C]dx\right|, \tag{31}$$

where $C \in R$ is a constant.

If we choose $C = \gamma_{2m}$, we have

$$\left|I - F_{2m} - \frac{(b-a)^{2m+1}}{(2m)!\,2^{2m}}\left(\frac{1}{2m+1} - \theta\right)\frac{f^{(2m-1)}(b) - f^{(2m-1)}(a)}{b-a}\right|$$
$$\leq \max_{x\in[a,b]}\left|G_{2m}(x) - \frac{1}{b-a}\int_a^b G_{2m}(x)dx\right|\int_a^b |f^{(2m)}(x) - \gamma_{2m}|dx, \tag{32}$$

and hence inequality (26) follows from (32) and (16).

Similarly we can prove that inequality (27) holds. □

Remark 4. If n is an odd integer and we take $\theta = 1$ in (23), (24) and (25), respectively, we recapture inequalities (2), (4) and (5), respectively. If n is an odd integer and we take $\theta = 0$ in (23), we recapture inequality (7). If n is an odd integer and we take $\theta = \frac{1}{3}$ in (23), (24) and (25), respectively, we recapture [15, inequality (16)], [15, inequality (17)] and [15, inequality (18)], respectively. If n is an even integer and we take $\theta = \frac{1}{3}$ in (26) and (27), respectively, we recapture [15, inequality (19)] and [15, inequality (20)], respectively. For other special cases, such as $\theta = 0$ or $\theta = \frac{1}{2}$, the interested reader can get some new error bounds for other quadrature rules, which we omit here.

Finally, we derive two sharp error inequalities when n is an odd and an even integer, respectively.

Theorem 8. *Let $f : [a, b] \to \mathbb{R}$ be a function such that $f^{(n-1)}$ is absolutely continuous on $[a, b]$ and $f^{(n)} \in L^2[a, b]$, where $n \geq 1$ is an odd integer. Then we have*

$$|I - F_n| \leq \frac{(b-a)^{n+\frac{1}{2}}}{n!\, 2^n} \sqrt{\frac{\theta^2 n^2(2n+1) - \theta(4n^2-1) + (2n-1)}{(2n+1)(2n-1)}} \sqrt{\sigma(f^{(n)})}, \tag{33}$$

for all $\theta \in [0, 1]$, where $\sigma(\cdot)$ is defined by $\sigma(f) = \|f\|_2^2 - \frac{1}{b-a}\left(\int_a^b f(x)dx\right)^2$. Inequality (33) is sharp in the sense that the constant $\frac{1}{n!\,2^n}\sqrt{\frac{\theta^2 n^2(2n+1)-\theta(4n^2-1)+(2n-1)}{(2n+1)(2n-1)}}$ cannot be replaced by a smaller one.

Proof. From (10), (12) and (15), we can easily get

$$\begin{aligned}
|I - F_n| &= \left| \int_a^b G_n(x) \left[f^{(n)}(x) - \frac{1}{b-a}\int_a^b f^{(n)}(x)dx \right] dx \right| \\
&\leq \left(\int_a^b G_n^2(x)dx \right)^{\frac{1}{2}} \left(\int_a^b \left[f^{(n)}(x) - \frac{1}{b-a}\int_a^b f^{(n)}(x)dx \right]^2 dx \right)^{\frac{1}{2}} \\
&= \left(\frac{[\theta^2 n^2(2n+1) - \theta(4n^2-1) + (2n-1)](b-a)^{2n+1}}{(2n+1)(2n-1)(n!)^2 2^{2n}} \right)^{\frac{1}{2}} \\
&\quad \times \left(\|f^{(n)}\|_2^2 - \frac{[f^{(n-1)}(b) - f^{(n-1)}(a)]^2}{b-a} \right)^{\frac{1}{2}} \\
&= \frac{(b-a)^{n+\frac{1}{2}}}{n!\, 2^n} \sqrt{\frac{\theta^2 n^2(2n+1) - \theta(4n^2-1) + (2n-1)}{(2n+1)(2n-1)}} \sqrt{\sigma(f^{(n)})}.
\end{aligned}$$

To prove the sharpness of (33), we suppose that (33) holds with a constant $C > 0$ as

$$|I - F_n| \leq C(b-a)^{n+\frac{1}{2}} \sqrt{\sigma(f^{(n)})}. \tag{34}$$

We may find a function $f : [a, b] \to \mathbb{R}$ such that $f^{(n-1)}$ is absolutely continuous on $[a, b]$ as

$$f^{(n-1)}(x) = \begin{cases} \dfrac{(x-a)^n}{(n+1)!}\left[x - a - \dfrac{\theta(n+1)(b-a)}{2} \right], & x \in \left[a, \dfrac{a+b}{2}\right], \\ \dfrac{(x-b)^n}{(n+1)!}\left[x - b + \dfrac{\theta(n+1)(b-a)}{2} \right], & x \in \left(\dfrac{a+b}{2}, b\right]. \end{cases}$$

It follows that

$$f^{(n)}(x) = G_n(x). \tag{35}$$

It's easy to find that the left-hand side of inequality (34) becomes

$$L.H.S.(34) = \frac{[\theta^2 n^2(2n+1) - \theta(4n^2-1) + (2n-1)](b-a)^{2n+1}}{(2n+1)(2n-1)(n!)^2 2^{2n}}, \tag{36}$$

and the right-hand side of inequality (34) is

$$R.H.S.(34) = \frac{1}{n!\, 2^n} \sqrt{\frac{\theta^2 n^2(2n+1) - \theta(4n^2-1) + (2n-1)}{(2n+1)(2n-1)}} C(b-a)^{2n+1}. \tag{37}$$

It follows from (34), (36) and (37) that

$$C \geq \frac{1}{n!\, 2^n} \sqrt{\frac{\theta^2 n^2(2n+1) - \theta(4n^2-1) + (2n-1)}{(2n+1)(2n-1)}},$$

which prove that the constant $\frac{1}{n!\,2^n}\sqrt{\frac{\theta^2 n^2(2n+1)-\theta(4n^2-1)+(2n-1)}{(2n+1)(2n-1)}}$ is the best possible in (33). $\square$

Theorem 9. *Let $f : [a,b] \to \mathbb{R}$ be a function such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$ and $f^{(n)} \in L^2[a,b]$, where $n > 1$ is an even integer ($n = 2m$). Then we have*

$$\begin{aligned}
&\left| I - F_{2m} - \frac{(b-a)^{2m+1}}{(2m)!\,2^{2m}} \left(\frac{1}{2m+1} - \theta \right) \frac{f^{(2m-1)}(b) - f^{(2m-1)}(a)}{b-a} \right| \\
\leq &\frac{(b-a)^{2m+\frac{1}{2}}}{(2m)!\,2^{2m}} \sqrt{\frac{[\theta^2(2m)^2(4m+1) - \theta(16m^2-1) + (4m-1)] - (16m^2-1)(\frac{1}{2m+1} - \theta)^2}{(4m+1)(4m-1)}} \sqrt{\sigma(f^{(2m)})},
\end{aligned} \tag{38}$$

for all $\theta \in [0,1]$. Inequality (38) is sharp in the sense that the constant

$$\frac{1}{(2m)!\,2^{2m}} \sqrt{\frac{[\theta^2(2m)^2(4m+1) - \theta(16m^2-1) + (4m-1)] - (16m^2-1)(\frac{1}{2m+1} - \theta)^2}{(4m+1)(4m-1)}}$$

cannot be replaced by a smaller one.

Proof. From (10), (12) and (15), we can easily get

$$\begin{aligned}
&\left| I - F_{2m} - \frac{(b-a)^{2m+1}}{(2m)!\,2^{2m}} \left(\frac{1}{2m+1} - \theta \right) \frac{f^{(2m-1)}(b) - f^{(2m-1)}(a)}{b-a} \right| \\
= &\left| \int_a^b G_{2m}(x) f^{(2m)}(x)dx - \frac{1}{b-a} \int_a^b G_{2m}(x)dx \int_a^b f^{(2m)}(x)dx \right| \\
= &\frac{1}{2(b-a)} \left| \int_a^b \int_a^b [G_{2m}(x) - G_{2m}(t)][f^{(2m)}(x) - f^{(2m)}(t)]dxdt \right| \\
\leq &\frac{1}{2(b-a)} \left(\int_a^b \int_a^b [G_{2m}(x) - G_{2m}(t)]^2 dxdt \right)^{\frac{1}{2}} \left(\int_a^b \int_a^b [f^{(2m)}(x) - f^{(2m)}(t)]^2 dxdt \right)^{\frac{1}{2}} \\
= &\left(\int_a^b G_{2m}^2(x)dx - \frac{1}{b-a} \left[\int_a^b G_{2m}(t)dt \right]^2 \right)^{\frac{1}{2}} \left(\int_a^b [f^{(2m)}(x)]^2 dx - \frac{1}{b-a} \left[\int_a^b f^{(2m)}(t)dt \right]^2 \right)^{\frac{1}{2}} \\
= &\frac{(b-a)^{2m+\frac{1}{2}}}{(2m)!\,2^{2m}} \sqrt{\frac{[\theta^2(2m)^2(4m+1) - \theta(16m^2-1) + (4m-1)] - (16m^2-1)(\frac{1}{2m+1} - \theta)^2}{(4m+1)(4m-1)}} \sqrt{\sigma(f^{(2m)})}.
\end{aligned}$$

To prove the sharpness of (38), we suppose that (38) holds with a constant $C > 0$ as

$$\begin{aligned}
&\left| I - F_{2m} - \frac{(b-a)^{2m+1}}{(2m)!\,2^{2m}} \left(\frac{1}{2m+1} - \theta \right) \frac{f^{(2m-1)}(b) - f^{(2m-1)}(a)}{b-a} \right| \\
\leq &C(b-a)^{2m+\frac{1}{2}} \sqrt{\sigma(f^{(2m)})}.
\end{aligned} \tag{39}$$

We may find a function $f : [a,b] \to R$ such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$ as

$$\begin{aligned}
&f^{(2m-1)}(x) \\
= &\begin{cases} \dfrac{(x-a)^{2m}}{(2m+1)!} \left[x - a - \dfrac{\theta(2m+1)(b-a)}{2} \right] - \dfrac{(b-a)^{2m+1}}{(2m)!\,2^{2m+1}} \left(\dfrac{1}{2m+1} - \theta \right), & x \in \left[a, \dfrac{a+b}{2} \right], \\ \dfrac{(x-b)^{2m}}{(2m+1)!} \left[x - b + \dfrac{\theta(2m+1)(b-a)}{2} \right] + \dfrac{(b-a)^{2m+1}}{(2m)!\,2^{2m+1}} \left(\dfrac{1}{2m+1} - \theta \right), & x \in \left(\dfrac{a+b}{2}, b \right]. \end{cases}
\end{aligned}$$

It follows that

$$f^{(2m)}(x) = G_{2m}(x). \tag{40}$$

It's easy to find that the left-hand side of inequality (39) becomes

$$\begin{aligned}
&L.H.S.(39) \\
= &\frac{[\theta^2(2m)^2(4m+1) - \theta(16m^2-1) + (4m-1)] - (16m^2-1)(\frac{1}{2m+1} - \theta)^2}{(4m+1)(4m-1)((2m)!)^2 2^{4m}} (b-a)^{4m+1},
\end{aligned} \tag{41}$$

and the right-hand side of inequality (39) is

$$R.H.S.(39) = \frac{1}{(2m)!\,2^{2m}}\sqrt{\frac{[\theta^2(2m)^2(4m+1)-\theta(16m^2-1)+(4m-1)]-(16m^2-1)(\frac{1}{2m+1}-\theta)^2}{(4m+1)(4m-1)}}C(b-a)^{4m+1}. \tag{42}$$

It follows from (39), (41) and (42) that

$$C \geq \frac{1}{(2m)!\,2^{2m}}\sqrt{\frac{[\theta^2(2m)^2(4m+1)-\theta(16m^2-1)+(4m-1)]-(16m^2-1)(\frac{1}{2m+1}-\theta)^2}{(4m+1)(4m-1)}},$$

which prove that the constant $\frac{1}{(2m)!\,2^{2m}}\sqrt{\frac{[\theta^2(2m)^2(4m+1)-\theta(16m^2-1)+(4m-1)]-(16m^2-1)(\frac{1}{2m+1}-\theta)^2}{(4m+1)(4m-1)}}$ is the best possible in (38). □

Remark 5. If we take $\theta = \frac{1}{3}$ in (33) and (38), respectively, we recapture sharp Simpson type inequalities [19, inequality (24)] and [19, inequality (29)], respectively. For other special cases, such as $\theta = 0$, $\theta = 1$ or $\theta = \frac{1}{2}$, the interested reader can get some new sharp error inequalities for other quadrature rules, which we omit here.

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